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A least-squares finite element method for the Navier–Stokes equations

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Abstract

The Navier–Stokes equations for flow in a plane are reformulated as a first-order system in terms of stress and stream functions. Solutions of this system are obtained by the least-squares finite element method. A feature of this approach is that the linearised system gives rise to a symmetric and positive-definite linear algebra problem at each Newton iteration. Care over handling the incompressibility term is needed to ensure good results are obtained.

Keywords: First-order; Least-squares; Finite elements; Conservation of mass; Navier–Stokes equations; Stress and stream functions

1. A first-order reformulation of the Navier–Stokes equations

1.1. The Navier–Stokes equations

The Navier–Stokes system of equations for an incompressible fluid in steady flow and on which no body forces act is

$$-\frac{1}{Re}\nabla^2\vec{u} + \vec{u}\cdot\nabla\vec{u} + \nabla p = 0 \text{ in } \Omega, \tag{1}$$

$$\nabla\cdot\vec{u} = 0 \text{ in } \Omega. \tag{2}$$

The enclosed flow boundary conditions for (1) and (2) are

$$\vec{u} = \vec{g} \text{ on } \Gamma, \tag{3}$$

$$\int_\Omega p \, \mathrm{d}\Omega = 0. \tag{4}$$

The quantity Re is the Reynolds number, which we define as being the inverse of the viscosity parameter v. This set of equations is non-linear, and the non-linear term comes to dominate for high values of the Reynolds number. Much effort has gone into obtaining finite-element solutions of these equations, see for example [11–13,18,23].

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1.2. The first-order least-squares finite element method

The first-order least-squares solution of a system of m equations in n unknowns holding over a region Ω is found by minimising the least-squares functional

$$\sum_{i=1}^{m} \|L_i U - f_i\|_0^2,$$

where L_i is a first-order differential operator and $U \in [H^1(\Omega)]^n$ satisfies appropriate boundary conditions. We look for continuous, piecewise differentiable finite element approximations U_h to the minimum U in a finite-dimensional subspace of $[H^1(\Omega)]^n$. Hence in obtaining solutions of the Navier–Stokes equations (1) and (2) it is necessary to recast the second-order system as a nonlinear first-order one. A number of such formulations have been considered in the literature. One example of such a formulation is the velocity–vorticity–pressure formulation, see [4]. Solutions of a backward facing step problem using this formulation can be found in [14] and the driven cavity problem is solved in [14–17]. Another example is the semi-linear velocity–vorticity–head formulation, see [2,3,16].

1.3. A first-order reformulation of the Navier–Stokes equations in terms of stress and stream functions

For a fluid of velocity $\vec{u} = (u_1,\ u_2)$ in a Cartesian coordinate frame with axes x and y

$$\nabla\vec{u} = \begin{pmatrix} \dfrac{\partial u_1}{\partial x} & \dfrac{\partial u_2}{\partial x} \\ \dfrac{\partial u_1}{\partial y} & \dfrac{\partial u_2}{\partial y} \end{pmatrix}.$$

Then (1) can be written explicitly as

$$-\frac{1}{Re}\left(\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2}\right) + \frac{\partial p}{\partial x} + u_1\frac{\partial u_1}{\partial x} + u_2\frac{\partial u_1}{\partial y} = 0,$$
$$-\frac{1}{Re}\left(\frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2}\right) + \frac{\partial p}{\partial y} + u_1\frac{\partial u_2}{\partial x} + u_2\frac{\partial u_2}{\partial y} = 0.$$

We introduce R, the Reynolds stress tensor [19], defined as

$$R = \begin{pmatrix} u_1^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{pmatrix}.$$

This matrix has a divergence with two components

$$\nabla\cdot R = \begin{pmatrix} 2u_1\dfrac{\partial u_1}{\partial x} + u_2\dfrac{\partial u_1}{\partial y} + u_1\dfrac{\partial u_2}{\partial y} \\ 2u_2\dfrac{\partial u_2}{\partial y} + u_1\dfrac{\partial u_2}{\partial x} + u_2\dfrac{\partial u_1}{\partial x} \end{pmatrix}.$$

For incompressible fluids, which satisfy Eq. (2), this can be simplified to

$$\nabla\cdot R = \begin{pmatrix} u_1\dfrac{\partial u_1}{\partial x} + u_2\dfrac{\partial u_1}{\partial y} \\ u_1\dfrac{\partial u_2}{\partial x} + u_2\dfrac{\partial u_2}{\partial y} \end{pmatrix} = \vec{u}\cdot\nabla\vec{u}.$$

Let d denote the deformation tensor

$$d = \frac{1}{2}\begin{pmatrix} 2\dfrac{\partial u_1}{\partial x} & \dfrac{\partial u_1}{\partial y} + \dfrac{\partial u_2}{\partial x} \\ \dfrac{\partial u_1}{\partial y} + \dfrac{\partial u_2}{\partial x} & 2\dfrac{\partial u_2}{\partial y} \end{pmatrix}.$$

Then introduce a tensor σ_R such that

$$\sigma_R = -pI + 2\nu d - R, \tag{5}$$

where I is the identity matrix. Given that (1) and (2) hold then

$$\nabla \cdot \sigma_R = 0.$$

We introduce a stress function ϕ defined so that

$$\sigma_R = \begin{pmatrix} \phi_{yy} & -\phi_{xy} \\ -\phi_{xy} & \phi_{xx} \end{pmatrix}$$

and a stream function ψ defined as

$$u_1 = \psi_y,$$
$$u_2 = -\psi_x.$$

We can write (5) in terms of ϕ and ψ as

$$\phi_{yy} = -p + 2\nu\psi_{xy} - \psi_y^2,$$
$$-\phi_{xy} = \nu(\psi_{yy} - \psi_{xx}) + \psi_x\psi_y,$$
$$\phi_{xx} = -p - 2\nu\psi_{xy} - \psi_x^2.$$

We eliminate the pressure p and make the substitutions

$$U_1 = \phi_x, \quad U_2 = \phi_y, \quad U_3 = \psi_x, \quad U_4 = \psi_y$$

to obtain

$$-\frac{\partial U_1}{\partial x} + \frac{\partial U_2}{\partial y} = 4\nu\frac{\partial U_3}{\partial y} + U_3^2 - U_4^2,$$
$$-\frac{\partial U_1}{\partial y} - \frac{\partial U_2}{\partial x} = 2\nu\left(\frac{\partial U_4}{\partial y} - \frac{\partial U_3}{\partial x}\right) + 2U_3U_4.$$

So the Navier–Stokes equations (1) and (2) can be written in first-order form as

$$-\frac{\partial U_1}{\partial x} + \frac{\partial U_2}{\partial y} - 2\nu\frac{\partial U_3}{\partial y} - 2\nu\frac{\partial U_4}{\partial x} - U_3^2 + U_4^2 = f_1, \tag{6}$$

$$\frac{\partial U_1}{\partial y} + \frac{\partial U_2}{\partial x} - 2\nu\frac{\partial U_3}{\partial x} + 2\nu\frac{\partial U_4}{\partial y} + 2U_3U_4 = f_2, \tag{7}$$

$$\frac{\partial U_1}{\partial y} - \frac{\partial U_2}{\partial x} = f_3, \tag{8}$$

$$2\nu\frac{\partial U_3}{\partial y} - 2\nu\frac{\partial U_4}{\partial x} = f_4. \tag{9}$$

Appropriate boundary conditions for this system of equations are as for the equivalent system for the Stokes equations, see [6,21]. In terms of these variables the enclosed flow conditions (3) are

$$U_3 = -g_2(x,y), \quad U_4 = g_1(x,y) \text{ on } \Gamma, \tag{10}$$

where $\vec{g} = (g_1, g_2)$. Given these conditions then the solution of the system (6)–(9) is unique provided that linear constraints $\Lambda_i(U) = 0$, $i = 1, \ldots, N_c$ are also specified. We fix both U_1 and U_2 at a point and either U_1 at a second point with a different x coordinate or U_2 at a second point with a different y coordinate, see [21].

Before moving on to develop a least-squares functional for the set of equations (6)–(9), we shall first linearise them.

A number of linearisation techniques have been employed in the finite element literature, as highlighted by Jiang in [15]. The one we shall use is Newton's linearisation method, see for example [2,3,15,17]. This is an iterative technique, with an updated solution $U^{\mathrm{T}} = (U_1, U_2, U_3, U_4)$ to be obtained from an estimate or previous iterate $\hat{U}^{\mathrm{T}} = (\hat{U}_1,\ \hat{U}_2,\ \hat{U}_3,\ \hat{U}_4)$. Applying Newton's linearisation method to the system (6)–(9) we obtain the equations

$$-\frac{\partial U_1}{\partial x} + \frac{\partial U_2}{\partial y} - 2v\frac{\partial U_3}{\partial y} - 2v\frac{\partial U_4}{\partial x} - 2U_3\hat{U}_3 + 2U_4\hat{U}_4 = f_1 - [\hat{U}_3]^2 + [\hat{U}_4]^2 \equiv f_1^*, \tag{11}$$

$$\frac{\partial U_1}{\partial y} + \frac{\partial U_2}{\partial x} - 2v\frac{\partial U_3}{\partial x} + 2v\frac{\partial U_4}{\partial y} + 2U_3\hat{U}_4 + 2U_4\hat{U}_3 = f_2 + 2\hat{U}_3\hat{U}_4 \equiv f_2^*, \tag{12}$$

$$\frac{\partial U_1}{\partial y} - \frac{\partial U_2}{\partial x} = f_3 \equiv f_3^*, \tag{13}$$

$$2v\frac{\partial U_3}{\partial y} - 2v\frac{\partial U_4}{\partial x} = f_4 \equiv f_4^*. \tag{14}$$

We write this system of equations as $L^*U = F^*$, with L^* a linear operator, U as above and $F^{*\mathrm{T}} = (f_1^*,\ f_2^*,\ f_3^*,\ f_4^*)$. The corresponding least-squares functional is

$$\sum_{i=1}^{4} \|L_i^* U - F_i^*\|_0^2 \tag{15}$$

with Ω the domain on which Eqs. (11)–(14) hold. We introduce a test space $\tilde{V}$ with elements V such that $V^{\mathrm{T}} = (V_1,\ V_2,\ V_3,\ V_4)$, namely

$$\tilde{V} = \{V \in [H^1(\Omega)]^4 | V_3 = 0, \quad V_4 = 0 \text{ on } \Gamma, \quad \Lambda_i(V) = 0,\ i = 1, \ldots, N_c\}.$$

We seek the function U, where $U^{\mathrm{T}} = (U_1,\ U_2,\ U_3,\ U_4)$, from an appropriate trial space $\tilde{U}$, namely

$$\tilde{U} = \left\{U \in [H^1(\Omega)]^4 | U_3 = -g_2, \quad U_4 = g_1 \text{ on } \Gamma, \quad \Lambda_i(U) = 0,\ i = 1, \ldots, N_c\right\},$$

such that the functional in Eq. (15) is a minimum. Given that the function U minimises the functional in Eq. (15) then

$$\lim_{t\to 0} \frac{d\int_\Omega (L^*U + tL^*V - F^*)^2 \mathrm{d}\Omega}{\mathrm{d}t} = 0 \quad \forall V \in \tilde{V}$$

and therefore

$$\int_\Omega L^*UL^*V \,\mathrm{d}\Omega = \int_\Omega L^*VF^* \,\mathrm{d}\Omega \quad \forall V \in \tilde{V}.$$

In obtaining finite element solutions we work with a finite-dimensional subset $\tilde{U}_h$ of the trial space $\tilde{U}$ and a finite-dimensional subset $\tilde{V}_h$ of the test space $\tilde{V}$. The finite element solution U_h satisfies the relation

$$\int_\Omega L^*U_hL^*V_h \,\mathrm{d}\Omega = \int_\Omega L^*V_hF^* \,\mathrm{d}\Omega \quad \forall V_h \in \tilde{V}_h.$$

In [5,6], we showed that mass is not conserved well in the solutions of the equivalent system for the Stokes equations for incompressible flow with enclosed flow boundary conditions. We saw in [5,6] that much more mass is conserved if we weight the terms in the least-squares functional corresponding to Eqs. (8) and (9). Although conservation of mass is enforced by Eq. (9), Eq. (8) is of the same form and it seems natural to weight this equation by the same factor. We note that loss of mass is observed in least-squares solutions of other first-order reformulations of the Stokes and Navier–Stokes equations and weighting of the mass conservation term or terms in these reformulations has been considered elsewhere, see for instance [8].

We have found that the terms corresponding to Eqs. (8) and (9) in the least-squares functional which arises from the Navier–Stokes system (6)–(9) also have to be weighted. Here, we present solutions obtained by minimising both the unweighted functional

$$S_N = \left\| -\frac{\partial U_1}{\partial x} + \frac{\partial U_2}{\partial y} - 2v\frac{\partial U_3}{\partial y} - 2v\frac{\partial U_4}{\partial x} - U_3^2 + U_4^2 - f_1 \right\|_0^2 + \left\| \frac{\partial U_1}{\partial y} + \frac{\partial U_2}{\partial x} - 2v\frac{\partial U_3}{\partial x} + 2v\frac{\partial U_4}{\partial y} + 2U_3U_4 - f_2 \right\|_0^2 + \left\| \frac{\partial U_1}{\partial y} - \frac{\partial U_2}{\partial x} - f_3 \right\|_0^2 + \left\| 2v\frac{\partial U_3}{\partial y} - 2v\frac{\partial U_4}{\partial x} - f_4 \right\|_0^2$$

and the weighted functional

$$S_{N,W} = \left\| -\frac{\partial U_1}{\partial x} + \frac{\partial U_2}{\partial y} - 2v\frac{\partial U_3}{\partial y} - 2v\frac{\partial U_4}{\partial x} - U_3^2 + U_4^2 - f_1 \right\|_0^2 + \left\| \frac{\partial U_1}{\partial y} + \frac{\partial U_2}{\partial x} - 2v\frac{\partial U_3}{\partial x} + 2v\frac{\partial U_4}{\partial y} + 2U_3U_4 - f_2 \right\|_0^2 + W\left\| \frac{\partial U_1}{\partial y} - \frac{\partial U_2}{\partial x} - f_3 \right\|_0^2 + W\left\| 2v\frac{\partial U_3}{\partial y} - 2v\frac{\partial U_4}{\partial x} - f_4 \right\|_0^2.$$

In this paper, we set $W = 10^3$ which is large enough so that there is relatively little flow lost but not so large that the linear algebra systems arising in the finite element solution are ill-conditioned, see [5]. Experience in solving the corresponding Stokes system suggests that the quality of solutions when weighting is employed is highly dependent on the form of the grid, see [6]. In approximating the Navier–Stokes equations, we use so-called "Union Jack" grids or grids which are topologically equivalent, in preference to other grids like those composed of unidirectional triangles. It is known that Union Jack grids possess special properties which do not necessarily hold for other forms of grid, see [20]. In particular they satisfy the grid decomposition property, see [1,9,10].

As a particular finite-dimensional space, we chose the set of piecewise continuous linear functions defined on a triangulation of Ω. In [7], convergence rates of least squares solutions of the first-order reformulation of the Stokes equations in terms of the velocity, vorticity and pressure and with enclosed flow boundary conditions are analysed. It is shown there that if linear elements are used to approximate all the variables then convergence is suboptimal in the vorticity and pressure. But the equivalent system to Eqs. (6)–(9) for the Stokes equations is the one in terms of the stress and stream functions presented in [21]. In [22], convergence rates of solutions of this system are analysed. Optimal convergence in H^1, and in L^2 given suitably smooth analytical solutions, is proved for a number of different boundary conditions including enclosed flow (10).

Local and global stiffness matrices and right-hand side vectors can be generated and assembled in the usual way to give a linear system for the unknown nodal values. As they arise from a least-squares functional, the linear systems are symmetric and positive-definite at each iteration.

The stream function can be recovered from the solution in the velocity variables by minimising the functional

$$S_s = \|\psi_x - U_3\|_0^2 + \|\psi_y - U_4\|_0^2.$$

2. Simulation of flow over a backward facing step

Our region is $[-2, 0] \times [-1, 0] \cup [0, 6] \times [-1, 1]$ as illustrated in Fig. 1.

On the inlet AB, we apply the boundary conditions

$$U_3 = 0, \quad U_4 = -y(1 + y),$$

whilst on the outlet CD, we apply the conditions

$$U_3 = 0, \quad U_4 = 0.125\left(1 - y^2\right).$$

On the walls BC, DE and AO, we apply the no-slip boundary conditions

$$U_3 = 0, \quad U_4 = 0.$$

The linear constraints are that $U_1 = 0$ and $U_2 = 0$ at the point B and that $U_2 = 0$ at the point D. The viscosity parameter v is set equal to 10^{-2}.

We see from Table 1 that there is substantial loss of mass in the solution of the unweighted S_N functional. For instance at $n_y = 4$ over 86% of the mass is lost between the inlet and the line $x = 0$ through the re-entrant

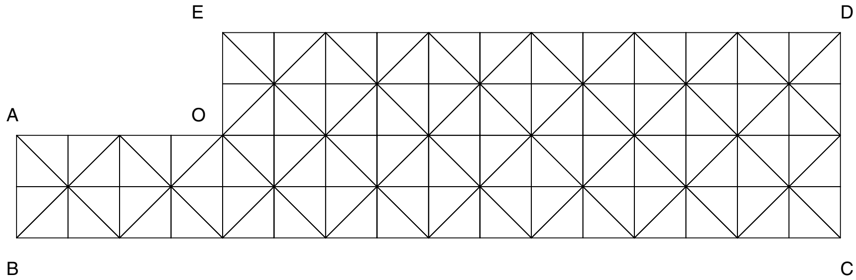

Fig. 1. Planar backward facing step grid at $n_y = 2$.

corner. Even at $n_y = 32$ almost 34% of the mass is lost between these two lines. Much less mass is lost in the solution of the weighted $S_{N,\ W}$ functional, see Table 2. In this case at $n_y = 32$ virtually all of the mass is conserved between the inlet and the line $x = 0$.

Figs. 2 and 3 show the contours of the stream functions obtained from the unweighted and weighted solutions respectively. Fig. 2 demonstrates graphically the loss of fluid in the solution, particularly in the area near the re-entrant corner. The solution shown in Fig. 3 is much more acceptable. There is some recirculation in the weighted solution close to the corner labelled E in Fig. 1. This is indicated in Fig. 3 by the closed contour lines in this portion of the region.

3. Flow over a semi-cylindrical restriction

We approximate flow through a channel [0, 10] × [0, 2.5] with a semi-circular restriction of radius unity, as illustrated in Fig. 4. The region is

$$\{(x, y) \mid x \in [0, 10], \quad y \in [0, 2.5], \quad (x - 3)^2 + y^2 \geqslant 1\}.$$

Our elements are triangular and the approximations are piecewise linear. The grid for the region with parameter $n_g = 1$ is shown in Fig. 4 and its first refinement, for which $n_g = 2$, is shown in Fig. 5. On the inlet line AF and the outlet line DE the boundary conditions are

$$U_3 = 0, \quad U_4 = 0.16y(2.5 - y).$$

The fluid is stationary on the walls AB, CD and EF as well as on the restriction itself so that

$$U_3 = 0, \quad U_4 = 0.$$

Table 1
Axial flow in the solution of the S_N formulation

n_y	Axial flow		
	$x = -2$	$x = 0$	$x = 3$
4	0.15625	0.02080	0.05201
8	0.16406	0.04288	0.07733
16	0.16602	0.07463	0.10480
32	0.16650	0.11067	0.13061

Table 2
Axial flow in the solution of the $S_{N,\ W}$ formulation, $W = 10^3$

n_y	Axial flow		
	$x = -2$	$x = 0$	$x = 3$
4	0.15625	0.14755	0.15503
8	0.16406	0.16171	0.16359
16	0.16602	0.16536	0.16585
32	0.16650	0.16632	0.16645

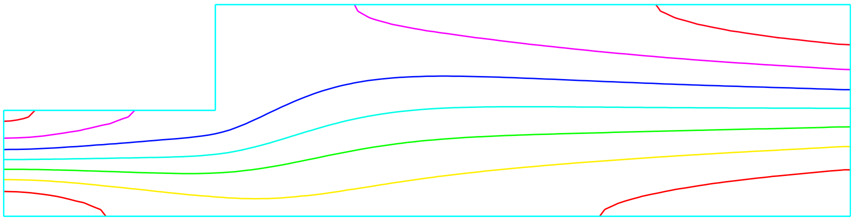

Fig. 2. Stream function contours, S_N formulation ($n_y = 16$).

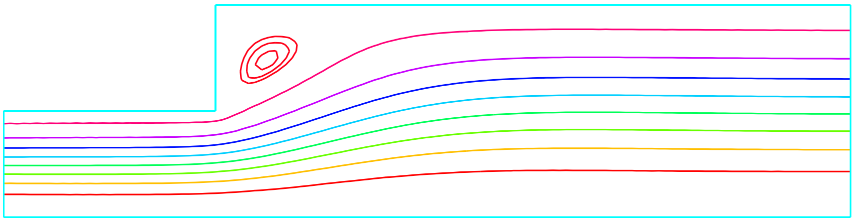

Fig. 3. Stream function contours, $S_{N,\,W}$ formulation ($n_y = 16$).

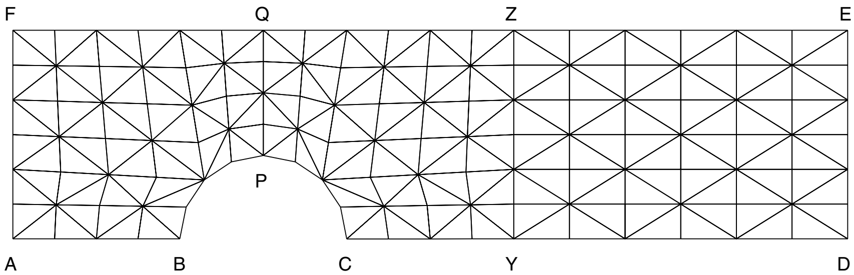

Fig. 4. Mesh at $n_g = 1$.

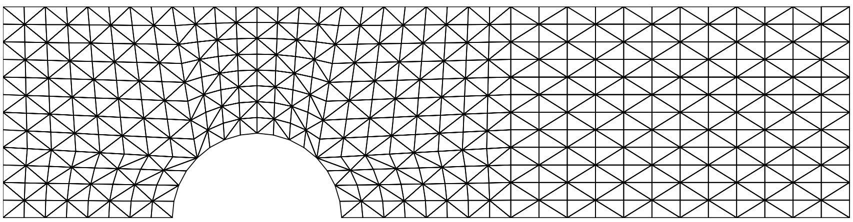

Fig. 5. Mesh at $n_g = 2$.

The viscosity parameter v is set equal to 10^{-2}. In this case, we find that we require three linear constraints. We set U_1 equal to zero at A and U_1 and U_2 as zero at E.

Just as in the solutions of the backward facing step problem there is a great deal of flow lost in the solution of the unweighted functional, see Table 3.

Table 3
Axial flow in the solution of the S_N formulation

n_g	Axial flow			
	AF	PQ	YZ	DE
4	0.41594	0.12013	0.18685	0.41594
8	0.41649	0.17901	0.23873	0.41649
16	0.41662	0.25224	0.29755	0.41662

Table 4
Axial flow in the solution of the $S_{N,W}$ formulation, $W = 10^3$

n_g	Axial flow			
	AF	PQ	YZ	DE
4	0.41594	0.38965	0.39807	0.41594
8	0.41649	0.41003	0.41212	0.41649
16	0.41662	0.41515	0.41563	0.41662

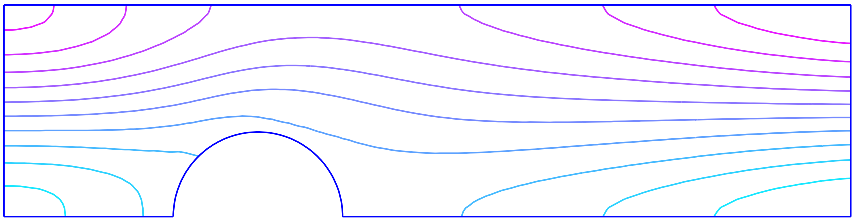

Fig. 6. Stream function contours, S_N formulation ($n_g = 4$).

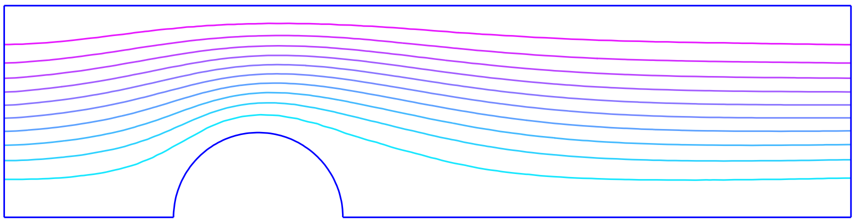

Fig. 7. Stream function contours, $S_{N,W}$ formulation ($n_g = 4$).

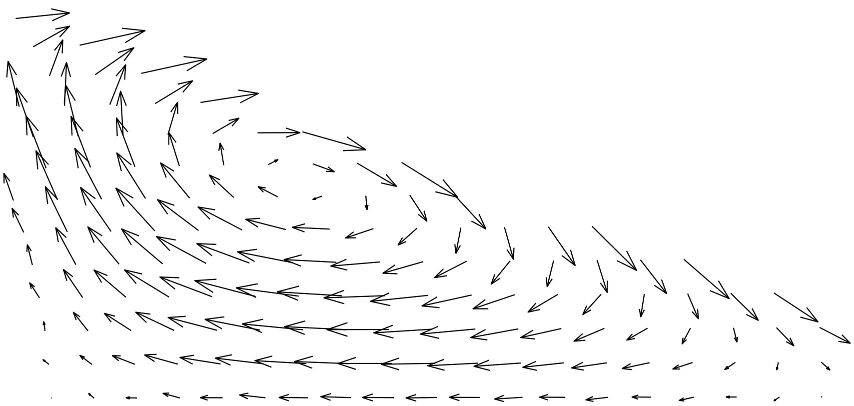

Fig. 8. Recirculation in the solution of the $S_{N,W}$ formulation, at $n_g = 8$.

At $n_g = 4$ even in the solution of the weighted functional 6.3% of the flow is lost between the inlet line AF and PQ, see Table 4. It is only in solutions of the $S_{N,\ W}$ functional on the finer grids that the flow is conserved.

The stream functions for the unweighted and weighted solutions at $n_g = 4$ are shown in Figs. 6 and 7, respectively. The diverging contours in Fig. 6 indicate a loss of flow, whilst the plot in Fig. 7 is closer to what we might expect for the true stream function. At this level of refinement there is no separation or recirculation visible even in the solution of the $S_{N,\ W}$ functional. There is some recirculation in the solution of the $S_{N,\ W}$ functional at $n_g = 8$ and $n_g = 16$ in the region to the immediate right of the cylinder, close to the corner labelled C in Fig. 4. A quiver plot of the velocity field in the solution on the grid $n_g = 8$ in this portion of the region is shown in Fig. 8.

4. Conclusion

The least-squares finite element method offers much promise because it gives rise to symmetric and positive-definite systems for which fast direct and indirect methods of solution exist. Although much work has been done, efficient solution techniques for standard Galerkin formulations of mixed methods for the Navier–Stokes equations are still being developed. In the formulation presented here the non-linear terms are algebraic and hence the system is semi-linear. This is similar to the formulation in terms of the velocity, the vorticity and the total pressure or head discussed for instance in [3]. We note that for these semi-linear systems the classification is the same as for the corresponding linear Stokes-equivalent systems so that the equations are elliptic for all values of the Reynolds number. Furthermore boundary conditions which are appropriate for the Stokes-equivalent system will also be appropriate for the system equivalent to the Navier–Stokes equations. The analysis of semi-linear systems may also be simpler. Other first-order formulations of the Navier–Stokes equations used in obtaining least-squares solutions do not possess this advantageous property.

We have shown here that apparent problems with lack of mass conservation in solutions of incompressible flow obtained using this formulation can be overcome by modifying the technique in a simple way, namely weighting of particular terms in the least-squares functional. This modication preserves the symmetric positive-definiteness of the linear systems that occur when using this method. Finally we note that the formulation can also be extended into three dimensions, see [5], although this idea is still being developed.

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